Analysis of the pure logic of necessitation and its extensions

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A PDF is available!

This slide is available online at:

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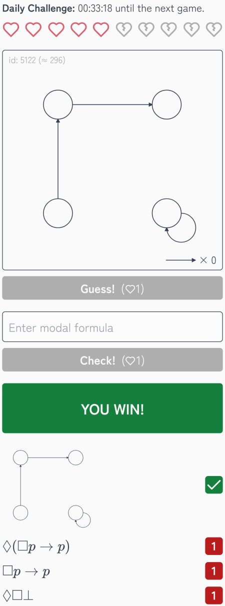
Bonus: the Kripke game!



I made a Wordle-like game where you guess the shape of a Kripke frame, just with formulas. Give it a try!

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What is The Pure Logic of Necessitation $\mathbf{N}$?

$\mathbf{N}$, the pure logic of necessitation

$\mathbf{N}$ is obtained from $\mathbf{K}$ by removing the K axiom

- or from the classical propositional logic by adding the necessitation rule ($\frac{\varphi}{\Box\varphi}$)

It was first introduced by Fitting et al. (1992)

- and they called it the *pure logic of necessitation*

It is a non-normal modal logic

- without congruence! ($\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$)
- so it doesn't have a neighborhood semantics
- instead, it has a Kripke-like semantics

The rationale of N (1)

Fitting et al. (1992) read $\Box\varphi$ in $\mathbf{N}$ as "φ is already derived"

- We cannot say ψ is *already* derived even if φ and $\varphi \to \psi$ have been derived!
- This justifies the lack of the K axiom: $\Box\varphi \land \Box(\varphi \to \psi) \to \Box\psi$
- They used $\mathbf{N}$ to analyze non-monotonic reasoning

The rationale of N (2)

Kurahashi (2024) considered $\Box$ in $\mathbf{N}$ the simplest notion of provability, in terms of provability logic

- The most fundamental property of provability should be: "if something is proved, then it is provable"
- This justifies the presence of the necessitation rule: $\frac{\varphi}{\Box\varphi}$
- He identified that $\mathbf{N}$ is exactly the provability logic of all provability predicates

The Kripke-like semantics for N

Without the K axiom, distinct $\Box$-formulas are hardly related

➡ The truth of $\Box\varphi$ must rely on its own accessibility relation

Definition (Fitting et al. (1992))

- Let $\mathscr{L}_\Box$ be the set of all modal formulas $(\bot, \wedge, \vee, \to, \Box)$
- $\mathfrak{F} = (W, \{\prec_\alpha\}_{\alpha \in \mathscr{L}_\Box})$ is an $\mathbf{N}$-*frame*
  $:\Longleftrightarrow$ $W \neq \emptyset$, and $\prec_\alpha \subseteq W \times W$ for each $\alpha \in \mathscr{L}_\Box$
- $\mathfrak{M} = (\mathfrak{F}, \Vdash)$ is an $\mathbf{N}$-*model*
  $:\Longleftrightarrow$ $\mathfrak{F}$ is an $\mathbf{N}$-frame, and $\Vdash$ is a valuation:
  - $w \Vdash \Box\varphi \;:\Longleftrightarrow\; w' \Vdash \varphi$ for every $w' \in W$ s.t. $w \prec_\varphi w'$

Almost the same as Kripke semantics, with a twist on accessibility

Basic properties of $\mathbf{N}$

Theorem (Fitting et al. (1992))

$\mathbf{N}$ has the finite frame property (FFP) w.r.t. all $\mathbf{N}$-frames

Proof.

Routine, by constructing a finite model of $\mathbf{N}$. $\square$

Proposition

$\mathbf{N}$ is not locally tabular

Proof.

We have an infinite sequence of provably distinct formulas:

$$\Box p,\ \Box\neg\neg p,\ \Box\neg^4 p,\ \Box\neg^6 p,\ \ldots$$

$\square$

Extending N with an Axiom

$\Box^n\varphi \to \Box^m\varphi$

Several extensions of $\mathbf{N}$

Kurahashi considered several extensions that have a direct application in provability logic:

Theorem (Kurahashi (2024))

- $\mathbf{N4} := \mathbf{N} + \Box\varphi \to \Box\Box\varphi$ has FFP w.r.t. transitive $\mathbf{N}$-frames:
$$x \prec_{\Box\varphi} y \;\&\; y \prec_{\varphi} z \implies x \prec_{\varphi} z$$
- $\mathbf{NR} := \mathbf{N} + \frac{\neg\varphi}{\neg\Box\varphi}$ has FPP w.r.t. serial $\mathbf{N}$-frames:
$$\exists y\,(x \prec_{\varphi} y)$$

Logics over $\mathbf{N}$ are determined by a frame condition

➡ We can think of various $\mathbf{N}$ counterparts of normal modal logics!

➡ Let's begin with generalizing the transitivity axiom

$\mathrm{Acc}_{m,n}$, the generalized transitivity axiom

Definition

- We write $x \prec^k_\varphi y$ to mean that there are $w_{k-1}, \dots, w_1$ s.t.:

  $$x \prec_{\Box^{k-1}\varphi} w_{k-1} \prec_{\Box^{k-2}\varphi} w_{k-2} \cdots w_2 \prec_{\Box\varphi} w_1 \prec_\varphi y$$
- (m, n)-accessibility is: $x \prec^m_\psi y \implies x \prec^n_\psi y$
  - transitivity is just $(2, 1)$-accessibility
- $\mathrm{Acc}_{m,n} \ := \ \Box^n\varphi \to \Box^m\varphi$
  - $\mathbf{N4}$ is exactly $\mathbf{N} + \mathrm{Acc}_{2,1}$

Now one may wonder:

Problem

Does $\mathbf{N} + \mathrm{Acc}_{m,n}$ have FFP w.r.t. (m, n)-accessible $\mathbf{N}$-frames?

However, $\mathbf{N} + \mathrm{Acc}_{m,n}$ is not complete for some $m, n \in \mathbb{N}$:

Proposition

For $n \geq 2$, (1) $\neg\Box^{n+1}\bot$ is valid in all $(0, n)$-accessible $\mathbf{N}$-frames, but (2) $\mathbf{N} + \mathrm{Acc}_{0,n} \nvdash \neg\Box^{n+1}\bot$

Proof.

(1) Easy. (2) One can actually construct an $\mathbf{N}$-model where $\mathrm{Acc}_{0,n}$ is valid but $\neg\Box^{n+1}\bot$ is not. $\Box$

$\mathbf{N}$-models allow more subtle construction of countermodels as the accessibility relation $\prec_\alpha$ can be tweaked for each α!

An additional rule to the rescue

Here, $\neg\Box^n\bot$ is provable in $\mathbf{N} + \mathrm{Acc}_{0,n}$ but $\neg\Box^{n+1}\bot$ is not

➡ so adding the following rule would recover completeness:

$$\frac{\neg\Box\varphi}{\neg\Box\Box\varphi}$$

Proposition

This rule is admissible in every normal modal logic

Corollary

$\mathbf{N} + \mathrm{Acc}_{m,n} \;\subseteq\; \mathbf{N} + \frac{\neg\Box\varphi}{\neg\Box\Box\varphi} + \mathrm{Acc}_{m,n} \;\subseteq\; \mathbf{K} + \mathrm{Acc}_{m,n}$

The finite frame property of $\mathbf{N} + \frac{\neg\Box\varphi}{\neg\Box\Box\varphi} + \mathbf{Acc}_{m,n}$

Definition

$\mathbf{NA}_{m,n} := \mathbf{N} + \mathrm{Acc}_{m,n}$, and $\mathbf{N}^{+}\mathbf{A}_{m,n} := \mathbf{N} + \frac{\neg\Box\varphi}{\neg\Box\Box\varphi} + \mathrm{Acc}_{m,n}$

Theorem (K. & S.)

$\mathbf{N}^{+}\mathbf{A}_{m,n}$ has FFP w.r.t. (m, n)-accessible $\mathbf{N}$-frames

Proof.

We carefully construct a finite (m, n)-accessible countermodel for a non-theorem of $\mathbf{N}^{+}\mathbf{A}_{m,n}$. We note that the presence of $\frac{\neg\Box\varphi}{\neg\Box\Box\varphi}$ indeed contributes to the construction. $\Box$

Interpolation properties in $\mathrm{NA}_{m,n}$ and $\mathrm{N}^{+}\mathrm{A}_{m,n}$ (1)

The rule $\frac{\neg\Box\varphi}{\neg\Box\Box\varphi}$ seems to be only relevant when we consider the completeness theory w.r.t. the Kripke-like semantics.

The interpolation theorems hold with or without the rule:

Proposition

Both $\mathbf{NA}_{m,n}$ and $\mathbf{N}^{+}\mathbf{A}_{m,n}$ have cut-admissible sequent calculi

Corollary

Both $\mathbf{NA}_{m,n}$ and $\mathbf{N}^{+}\mathbf{A}_{m,n}$ enjoy CIP and LIP

Proof.

Just Use Maehara's Method™ $\square$

Interpolation properties in $\mathrm{NA}_{m,n}$ and $\mathrm{N}^{+}\mathrm{A}_{m,n}$ (2)

We obtained an even stronger result:

Theorem

Both $\mathbf{NA}_{m,n}$ and $\mathbf{N}^{+}\mathbf{A}_{m,n}$ enjoy ULIP

Proof.

We embed both logics to the classical propositional logic $\mathbf{Cl}$, and reduce the problem to ULIP of $\mathbf{Cl}$, which is known. $\square$

Here, ULIP (uniform Lyndon —) is a strengthening of both UIP and LIP. See Kurahashi (2020) for details.

Bonus: a general method for proving ULIP

We also developed a general method for proving ULIP:

Theorem (S.)

For any logics $L \subseteq M$, if there is an embedding of M into L with certain properties, and L has ULIP, then so does M

Example

By the double negation embedding, ULIP of the intuitionistic propositional logic $\mathbf{Int}$ implies ULIP of $\mathbf{Cl}$.

No deep dive today. See Sato (2025) for details!

The Showdown (vs. $\mathbf{K} + \Box^n \varphi \to \Box^m \varphi$)

The showdown

Recall that:

$$\mathbf{NA}_{m,n} \subseteq \mathbf{N}^{+}\mathbf{A}_{m,n} \subseteq \mathbf{K} + \mathrm{Acc}_{m,n}$$

We shall compare the following properties of the above logics, which would highlight intriguing differences between them:

- Completeness
- The finite frame property
- The interpolation properties (CIP, LIP, UIP, ULIP)

Completeness: the hidden gems?

There is a classic result by Lemmon & Scott that $\mathbf{K} + \mathrm{Acc}_{m,n}$ is complete for every $m, n \in \mathbb{N}$

So it is interesting that $\mathbf{NA}_{m,n}$ is incomplete for some cases, and requires an additional rule $\frac{\neg\Box\varphi}{\neg\Box\Box\varphi}$ to fix it

- This rule is admissible in most logics, but seems to be very important for any logic with the necessitation rule

Open Problem

Is there any other rule that is admissible in normal modal logics, but is essential for completeness of some logic extending $\mathbf{N}$?

The finite frame property: why so hard?

FFP of $\mathbf{K} + \mathrm{Acc}_{m,n}$ has been left unsolved for decades, especially when $m < n$:

- Zakharyaschev (1997) referred to it as "one of the major challenges in completeness theory"
- the cases when $m \geq 0$, $n = 1$ are solved by Gabbay (1972)

On the other hand, FFP of $\mathbf{N}^+\mathbf{A}_{m,n}$ is obtained, although not easily, by a direct construction of a finite countermodel!

Open Problem

Why is FFP of $\mathbf{K} + \mathrm{Acc}_{m,n}$ so hard to prove? Is there some logic between $\mathbf{N}^+\mathbf{A}_{m,n}$ and $\mathbf{K} + \mathrm{Acc}_{m,n}$ with the same difficulty?

Interpolation properties: the K axiom to blame?

It is known that $\mathbf{K} + \mathrm{Acc}_{m,n}$ does not, in general, enjoy all of CIP, LIP, UIP, and ULIP:

- Bílková (2007) proved that $\mathbf{K4} = \mathbf{K} + \mathrm{Acc}_{2,1}$ lacks UIP
- Marx (1995) proved that $\mathbf{K} + \mathrm{Acc}_{1,2}$ lacks even CIP!

However, for any m, n, $\mathbf{NA}_{m,n}$ and $\mathbf{N}^{+}\mathbf{A}_{m,n}$ enjoy all of them!

Open Problem

To what extent the presence of the K axiom is *harmful* for a logic in terms of interpolation properties?

- Is there a logic between $\mathbf{N4}$ and $\mathbf{K4}$ that lacks UIP?
- Is there a logic between $\mathbf{N} + \mathrm{Acc}_{1,2}$ and $\mathbf{K} + \mathrm{Acc}_{1,2}$ that lacks CIP?

That’s all!

Appendix & References

The propositionalization method (1/2)

Propositionalization is a method that can be used to reduce ULIP of logic to that of a weaker one. It proceeds like this:

Given a logic X, let $\mathscr{L}_X$ designate the language of X.

Consider logics L and M s.t. $\mathscr{L}_L \subseteq \mathscr{L}_M$ and $L \subseteq M$.

Definition

Let L' be the same logic as L, but its propositional variables extended by adding a fresh one p_φ for every $\varphi \in \mathscr{L}_M$.

Definition

Let $\sigma : \mathscr{L}'_L \to \mathscr{L}_M$ be the substitution that replaces every p_φ with φ. It is easy to see that $L' \vdash \rho$ implies $M \vdash \sigma(\rho)$ for any $\rho \in \mathscr{L}'_L$.

The propositionalization method (2/2)

Definition

A pair of translations $\sharp, \flat : \mathscr{L}_M \to \mathscr{L}'_L$ is called a propositionalization of M into L if the following are met:

1. $M \vdash \varphi \to \psi$ implies $L' \vdash \varphi^\flat \to \psi^\sharp$;
2. $M \vdash \sigma(\varphi^\sharp) \to \varphi$ and $M \vdash \varphi \to \sigma(\varphi^\flat)$;
3. For $(\bullet, \circ) \in \{(+,-),(-,+)\}$ and $\natural \in \{\sharp, \flat\}$,
   $p \in v^\bullet(\varphi^\natural)$ implies $p \in v^\bullet(\varphi)$, and $p_\psi \in v^\bullet(\varphi^\natural)$ implies both $v^\bullet(\psi) \subseteq v^\bullet(\varphi)$ and $v^\circ(\psi) \subseteq v^\circ(\varphi)$.

Theorem (S.)

If L has ULIP, and there is a propositionalization of M into L, then M also has ULIP.

References

This talk is based on the papers indicated by $\star$.

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Omori and Skurt rediscovered the same logic as $\mathbf{N}$, namely $\mathbf{M}^{+}$ in their paper. They also gave a non-deterministic many-valued semantics for $\mathbf{N}$.