

Uniform Lyndon Interpolation for $\mathbf{N}^+ \mathbf{A}_{m,n}$

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The slides are available online at:

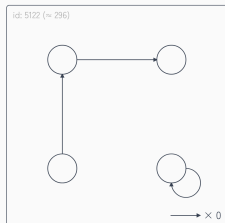
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(will be displayed again at the end)

Bonus: the Kripke game!

Daily Challenge: 00:33:18 until the next game.



Guess! (♡1)

Enter modal formula

Check! (♡1)

YOU WIN!



$\Diamond(\Box p \rightarrow p)$

$\Box p \rightarrow p$

$\Diamond\Box\perp$



I made a Wordle-like game
where you guess the shape of a
Kripke frame, just with formulas.
Give it a try!

cannorin.net/kripke



I proved that the logic $\mathbf{N}^+ \mathbf{A}_{m,n}$ enjoys
Uniform Lyndon interpolation property,
with a new method called propositionalization

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The Logic $N^+A_{m,n}$

What is $N^+A_{m,n}$?

$$N := Cl + \frac{\varphi}{\Box\varphi}$$

$$N^+A_{m,n} := N + \frac{\neg\Box\varphi}{\neg\Box\Box\varphi} + \Box^n\varphi \rightarrow \Box^m\varphi$$

- **Cl**: the classical propositional logic
- **N**: the pure logic of necessitation (Fitting et al. 1992)
- $\frac{\neg\Box\varphi}{\neg\Box\Box\varphi}$: required by the semantics*
- $\Box^n\varphi \rightarrow \Box^m\varphi$: a generalized reflexivity/transitivity axiom

*the completeness does not hold without it. no deep dive today

$N^+A_{m,n}$ vs. normal modal logics

Fact (Kurahashi and S.)

$$N^+A_{m,n} \subsetneq K + \Box^n \varphi \rightarrow \Box^m \varphi$$

Proof.

The rule $\frac{\neg \Box \varphi}{\neg \Box \Box \varphi}$ is admissible in K . The rest is trivial. \square

Fact (Kurahashi and S.)

$N^+A_{m,n}$ has the finite frame property (ffp) for every $m, n \in \mathbb{N}$

It is still unknown to this day whether $K + \Box^n \varphi \rightarrow \Box^m \varphi$ has ffp

➡ The lack of the K axiom is indeed a massive difference

The sequent calculus for $N^+A_{m,n}$

A sequent calculus $G_{N^+A_{m,n}}$ is obtained from **LK** by adding:

$$\frac{\Rightarrow \varphi}{\Rightarrow \Box \varphi} \text{ (nec)}$$

$$\frac{\Box \varphi \Rightarrow}{\Box \Box \varphi \Rightarrow} \text{ (rosbox, when } m = 0 \text{ and } n \geq 2)$$

$$\frac{\Box^m \varphi, \Box^n \varphi, \Gamma \Rightarrow \Delta}{\Box^n \varphi, \Gamma \Rightarrow \Delta} \text{ (accL, when } n > m)$$

$$\frac{\Gamma \Rightarrow \Delta, \Box^m \varphi, \Box^n \varphi}{\Gamma \Rightarrow \Delta, \Box^m \varphi} \text{ (accR, when } m > n)$$

Proposition (S.)

- $G_{N^+A_{m,n}}$ proves $\Gamma \Rightarrow \Delta$ iff $N^+A_{m,n}$ proves $\bigwedge \Gamma \rightarrow \bigvee \Delta$
- $G_{N^+A_{m,n}}$ admits cut elimination

Uniform Lyndon Interpolation Property

CIP and LIP (1/2)

Let $V^+(\varphi)$ and $V^-(\varphi)$ denote the set of variables that occur in φ positively and negatively, resp. Let also $V(\varphi) = V^+(\varphi) \cup V^-(\varphi)$

L is said to enjoy Craig interpolation property (CIP) if for every φ, ψ s.t. $L \vdash \varphi \rightarrow \psi$, there is χ s.t.:

1. $V(\chi) \subseteq V(\varphi) \cap V(\psi)$;
2. $L \vdash \varphi \rightarrow \chi$ and $L \vdash \chi \rightarrow \psi$.

Such χ is called an interpolant of $\varphi \rightarrow \psi$

CIP and LIP (2/2)

Let $V^+(\varphi)$ and $V^-(\varphi)$ denote the set of variables that occur in φ positively and negatively, resp. Let also $V(\varphi) = V^+(\varphi) \cup V^-(\varphi)$

L is said to enjoy Lyndon interpolation property (LIP) if for every φ, ψ s.t. $L \vdash \varphi \rightarrow \psi$, there is χ s.t.:

1. $V^\bullet(\chi) \subseteq V^\bullet(\varphi) \cap V^\bullet(\psi)$ ($\bullet \in \{+, -\}$);
2. $L \vdash \varphi \rightarrow \chi$ and $L \vdash \chi \rightarrow \psi$.

Such χ is called an interpolant of $\varphi \rightarrow \psi$

UIP and ULIP (1/2)

L is said to enjoy Uniform interpolation property (UIP) if for any φ and any finite set of variables P , there is χ s.t.

1. $V(\chi) \subseteq V(\varphi) \setminus P$;
2. $L \vdash \varphi \rightarrow \chi$;
3. $L \vdash \chi \rightarrow \psi$ for any ψ s.t. $L \vdash \varphi \rightarrow \psi$ and $V(\psi) \cap P = \emptyset$.

Such χ is called a post-interpolant of (φ, P)

UIP and ULIP (2/2)

L is said to enjoy Uniform Lyndon interpolation property (ULIP) if for any φ and any finite sets of variables P^+, P^- , there is χ s.t.

1. $V^\bullet(\chi) \subseteq V^\bullet(\varphi) \setminus P^\bullet$ ($\bullet \in \{+, -\}$);
2. $L \vdash \varphi \rightarrow \chi$;
3. $L \vdash \chi \rightarrow \psi$ for any ψ s.t. $L \vdash \varphi \rightarrow \psi$ and $V^\bullet(\psi) \cap P^\bullet = \emptyset$ ($\bullet \in \{+, -\}$).

Such χ is called a post-interpolant of (φ, P^+, P^-)

Several facts on the interpolation properties (1/2)

Fact

- If L has UIP, then L has CIP
- If L has LIP, then L has CIP
- If L has ULIP, then L has both UIP and LIP (Kurahashi 2020)

Fact (Kurahashi 2020)

- The classical propositional logic **CI** enjoys ULIP
- The modal logic **K** enjoys ULIP

Several facts on the interpolation properties (2/2)

The situation is complicated for the extensions of **K**:

Fact

- **KT** = **K** + $\Box\varphi \rightarrow \varphi$ enjoys ULIP (Kurahashi 2020)
- For $m > 0$, **K** + $\Box\varphi \rightarrow \Box^m\varphi$ enjoys CIP (Gabbay 1972) and LIP (Kuznets 2016)
- **K4** = **K** + $\Box\varphi \rightarrow \Box\Box\varphi$ lacks UIP (Bílková 2007)
- **K** + $\Box\Box\varphi \rightarrow \Box\varphi$ lacks even CIP (Marx 1995)

K + $\Box^n\varphi \rightarrow \Box^m\varphi$, in general, may or may not enjoy them

➔ What happens if we weaken it to $\mathbf{N}^+\mathbf{A}_{m,n}$?

The Propositionalization Method

Propositionalization, in short

ULIP of a logic is sometimes proven by embedding it to some weaker logic where ULIP is already known:

Example

Through the boxdot translation, ULIP of **K** implies ULIP of **KT**, and the failure of it in **S4** implies that of **K4**

I gave a sufficient condition on such embeddings:

Theorem (S.)

For any logics $L \subseteq M$, if there is a special embedding of M into L called propositionalization, and L has ULIP, then so does M

Propositionalization, in detail (1/3)

Given a logic X , let \mathcal{L}_X designate the language of X .

Consider logics L and M s.t. $\mathcal{L}_L \subseteq \mathcal{L}_M$ and $L \subseteq M$.

Now we want to *propositionalize* any \mathcal{L}_M -formula that is not expressible in \mathcal{L}_L :

Definition

Let L' be the same logic as L , but its propositional variables extended by adding a fresh one p_φ for every $\varphi \in \mathcal{L}_M$.

Definition

Let $\sigma : \mathcal{L}'_L \rightarrow \mathcal{L}_M$ be the substitution that replaces every p_φ with φ , then $L' \vdash \rho$ implies $M \vdash \sigma(\rho)$ for any $\rho \in \mathcal{L}'_L$.

Propositionalization, in detail (2/3)

Definition

A pair of translations $\sharp, \flat : \mathcal{L}_M \rightarrow \mathcal{L}'_L$ is called a propositionalization of M into L if the following are met:

(Emulatable) $M \vdash \varphi \rightarrow \psi$ implies $L' \vdash \varphi^\flat \rightarrow \psi^\sharp$;

(Invertible) $M \vdash \sigma(\varphi^\sharp) \rightarrow \varphi$ and $M \vdash \varphi \rightarrow \sigma(\varphi^\flat)$;

(Polarity-preserving) For $(\bullet, \circ) \in \{(+, -), (-, +)\}$, $\natural \in \{\sharp, \flat\}$:

- $p \in V^\bullet(\varphi^\natural)$ implies $p \in V^\bullet(\varphi)$;
- $p_\psi \in V^\bullet(\varphi^\natural)$ implies $V^\bullet(\psi) \subseteq V^\bullet(\varphi)$, $V^\circ(\psi) \subseteq V^\circ(\varphi)$.

Propositionalization, in detail (3/3)

Theorem (S.)

If L has ULIP, and there is a propositionalization of M into L , then M also has ULIP

Proof (outline).

Take any φ , P^+ , P^- . We extend P^\bullet to Q^\bullet by adding every $p_\psi \in V(\varphi^b)$ s.t. $V^\bullet(\psi) \cap P^\bullet \neq \emptyset$. By ULIP of L , we get a post-interpolant of (χ', Q^+, Q^-) . Then, emulatability, invertibility, and polarity-preservingness of \sharp, \flat assert that $\chi = \sigma(\chi')$ is indeed a post-interpolant of (φ, P^+, P^-) in M . \square

The Main Theorem

The Main Theorem

Theorem (S.)

There is a propositionalization (\sharp, \flat) of $\mathbf{N}^+ \mathbf{A}_{m,n}$ into \mathbf{Cl}

Proof (outline).

We construct such \sharp, \flat that each additional rule in $\mathbf{G}_{\mathbf{N}^+ \mathbf{A}_{m,n}}$ can be emulated in \mathbf{LK} : $\mathbf{G}_{\mathbf{N}^+ \mathbf{A}_{m,n}} \vdash \Gamma \Rightarrow \Delta$ implies $\mathbf{LK} \vdash \Gamma^\flat \Rightarrow \Delta^\sharp$. We also ensure invertibility and polarity-preservingness by adding just the right amount of information to enable emulation. \square

Corollary

$\mathbf{N}^+ \mathbf{A}_{m,n}$ enjoys ULIP!

Summing it up (1/2)

It is known that $\mathbf{K} + \Box^n \varphi \rightarrow \Box^m \varphi$ does not, in general, enjoy all of CIP, LIP, UIP, and ULIP:

- $\mathbf{K4} = \mathbf{K} + \Box \varphi \rightarrow \Box \Box \varphi$ lacks UIP
- $\mathbf{K} + \Box \Box \varphi \rightarrow \Box \varphi$ lacks even CIP

However, $\mathbf{N}^+ \mathbf{A}_{m,n}$ enjoy all of them for every $m, n \in \mathbb{N}$!

Summing it up (1/2)

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However, $\mathbf{N}^+ \mathbf{A}_{m,n}$ enjoy all of them for every $m, n \in \mathbb{N}$!

Open Problem

To what extent the presence of the \mathbf{K} axiom is *harmful* for a logic in terms of interpolation properties?

- Is there a logic between $\mathbf{N4}$ and $\mathbf{K4}$ that lacks UIP?
- Is there a logic between $\mathbf{N} + \Box \Box \varphi \rightarrow \Box \varphi$ and $\mathbf{K} + \Box \Box \varphi \rightarrow \Box \varphi$ that lacks CIP?

Summing it up (2/2)

We also developed a general method for proving ULIP:

Theorem (S.)

For any logics $L \subseteq M$, if there is an embedding of M into L with certain properties, and L has ULIP, then so does M

Summing it up (2/2)

We also developed a general method for proving ULIP:

Theorem (S.)

For any logics $L \subseteq M$, if there is an embedding of M into L with certain properties, and L has ULIP, then so does M

Open Problems

- Can we construct a propositionalization of **K** into **CI**?
- Can we characterize a syntactic property on sequent calculi that corresponds to the existence of a propositionalization? (e.g. Iemhoff 2019, Akbar Tabatabai & Jalali 2025)

Thanks!

That's all!

The Slides



cannorin.net/math/alc2025.pdf

The Kripke Game



cannorin.net/kripke

Appendix & References

References

This talk is based on the paper indicated by ★:

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